TIGHT CONTACT STRUCTURES ON LENS SPACES

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Abstract. In this paper we develop a method for studying tight contact structures on lens spaces. We then derive uniqueness and non-existence statements for tight contact structures with certain (half) Euler classes on lens spaces.

1. Introduction

Contact geometry has recently come to the foreground of low dimensional topology. Not only have there been striking advances in the understanding of contact structures on 3-manifolds [E4, Gi2, H], but there has been significant interplay with knot theory [R], symplectic geometry [LM, Et2], fluid dynamics [EG1], foliation theory [ET], and Seiberg-Witten theory [KM]. In 1971 Martinet [M] showed how to construct a contact structure on any 3-manifold. Several decades later it became clear that contact structures fell into two distinct classes: tight and overtwisted (see Section 2 for definitions). It is the tight contact structures that carry significant geometric information; where as, Eliashberg [E1] has shown that the understanding of overtwisted contact structures reduces to a "simple" algebraic question. Unfortunately, Martinet's theorem does not, in general, produce tight contact structures. The only general method for constructing tight structures is by Stein fillings [G, E2] or perturbing taut foliations [ET]. These techniques, however, leave the general existence question open. Even less is known concerning the uniqueness/classification question; it has only been answered on $S^3, S^1 \times S^2$ and $\mathbb{R}P^3$ [E4] and T^3 (and most T^2 bundles over S^1) [Gi2, K].

The purpose of this paper is to introduce some techniques for understanding tight contact structures. We apply them to the simplest class of 3-manifolds: lens spaces. Recall lens spaces are 3-manifolds that can be written as the union of two solid tori, or in other words, lens spaces are Heegaard genus one manifolds. On these manifolds we are able to derive some general uniqueness and non-existence statements

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in terms of the homotopy type of the contact structure. In particular, in Section 4.2 we show:

Theorem 4.8. On any lens space $L(p, q)$ there is at least one class in $H^2(L(p, q))$ realized by a unique tight contact structure and at least one class that cannot be realized by a tight contact structure.

It has been known for some time that on any 3-manifold there are only finitely many elements in second cohomology that can be realized by tight contact structures [E3]. This gives no restrictions for lens spaces since the second cohomology of a lens spaces is a finite group. More recently Kronheimer and Mrowka [KM] have shown that only finitely many homotopy types of plane fields can be realized by semi-fillable contact structures. Since any semi-fillable structure is tight and all currently known tight structures are semi-fillable, one is tempted to conjecture that this is also true for tight contact structures. This would restrict tight contact structures on lens spaces; it would not, however, say that there are a finite number of them. The techniques in this paper show:

Theorem 4.10. There are only finitely many tight contact structures (up to isotopy) on any lens space.

Moreover, on some lens spaces we can give a complete classification of contact structures.

Theorem 4.9. Classified up to isotopy:

1. *If $p = 0$ then $L(p, q) = S^1 \times S^2$ and there is a unique tight contact structure.*
2. *If $p = 1$ then $L(p, q) = S^3$ and there is a unique tight contact structure.*
3. *If $p = 2$ then $L(p, q) = \mathbb{R}P^3$ and there is a unique tight contact structure.*
4. *On $L(3, 1)$ there are exactly two tight contact structures (one for each non zero element in $H^2(L(3, 1); \mathbb{Z})$).*
5. *On $L(3, 2)$ there is exactly one tight contact structure (realizing the zero class in $H^2(L(3, 2))$).*

In future work we plan to push the analysis of tight contact structures on lens spaces further [1] as well as apply these techniques to 3-manifolds with higher Heegaard genus.

[1] Added in proof: E. Giroux and K. Honda (independently) have recently announced a complete classification of tight contact structures on lens spaces. Specifically they are all obtained from Stein fillings and determined by their half-Euler class.

2. Contact Structures In Three Dimensions

In this section we briefly recall some facts concerning contact geometry in dimension three. For more details see [A] or [E4].

Let M be an oriented 3-manifold. A *contact structure* ξ on M is a totally non-integrable 2-plane field in TM. We will only consider transversely orientable contact structures (this is not a serious restriction). This allows us to globally define the plane field ξ as the kernel of a 1-form α. Using the 1-form α, the Frobenius Theorem allows us to express the non-integrability of ξ as $\alpha \wedge d\alpha \neq 0$. Thus $\alpha \wedge d\alpha$ is a volume form on M. Other 1-forms we could use to define ξ will give us different volume forms but they will induce the same orientation on M (since any other 1-form α' that defines ξ must be of the form $f\alpha$ where $f : M \to \mathbb{R}$ is a non-zero function). Thus ξ orients M and we only consider contact structures ξ whose induced orientation agrees with the orientation on M. (A similar analysis could be done when the orientations disagree.)

Contact geometry presents interesting and difficult global problems; however, Darboux's Theorem tells us that all contact structures are *locally* contactomorphic. Two contact structures are *contactomorphic* if there is a diffeomorphism of the underlying manifolds that sends one of the plane fields to the other. Furthermore, Gray's Theorem tells us that a continuously varying family of contact structures are related by a continuously varying family of contactomorphisms. We have similar results near surfaces in M. If Σ is a surface in a contact manifold (M, ξ) then generically $T_p\Sigma \cap \xi_p$ will be a line in $T_p\Sigma$. Since a line field is always integrable $T_p\Sigma \cap \xi_p$ defines a natural singular foliation Σ_ξ associated to ξ called the *characteristic foliation.* As with Darboux's Theorem determining a contact structure in the neighborhood of a point, one can show that Σ_ξ determines the germ of ξ along Σ. The singular points of Σ_ξ occur where $\xi_p = T_p\Sigma$. Generically, singular points of Σ_ξ are either elliptic (if the local index is 1) or hyperbolic (if the local index is -1). Notice that if Σ is oriented then we can assign signs to the singular points of Σ_ξ. A singular point p is called *positive* if ξ_p and $T_p\Sigma$ have the same orientation, otherwise p is called *negative.* Moreover, the orientations on Σ and ξ "orient" the characteristic foliation, so we can think of the singular foliation as a flow. A very useful modification of Σ_ξ is given by the Elimination Lemma (proved in various forms by Giroux, Eliashberg and Fuchs, see [E5]).

Lemma 2.1 (Elimination Lemma)**.** *Let Σ be a surface in a contact* 3-*manifold (M, ξ). Assume that p is an elliptic and q is a hyperbolic singular point in Σ_ξ, they both have the same sign and there is a leaf*

γ in the characteristic foliation Σ_ξ that connects p to q. Then there is a C^0-small isotopy $\phi : \Sigma \times [0,1] \to M$ such that ϕ_0 is the inclusion map, ϕ_t is fixed on γ and outside any (arbitrarily small) pre-assigned neighborhood U of γ and $\Sigma' = \phi_1(\Sigma)$ has no singularities inside U.

There has recently emerged a fundamental dichotomy in three dimensional contact geometry. A contact structure ξ is called *overtwisted* if there exists an embedded disk D in M whose characteristic foliation D_ξ contains a limit cycle. If ξ is not overtwisted then it is called *tight*. Eliashberg [E1] has completely classified overtwisted contact structures on closed 3-manifolds: classifying overtwisted contact structures up to isotopy is equivalent to classifying plane fields up to homotopy (which has a purely algebraic solution). As discussed in the introduction, much less is known about tight contact structures. One of the main results about tight contact structures, on which all the results in this paper are based, is the following theorem of Eliashberg.

Theorem 2.2 (Eliashberg [E4], 1992)**.** *Two tight contact structures on the ball B^3 which induce the same characteristic foliations on ∂B^3 are isotopic relative to ∂B^3.*

A closed curve $\gamma : S^1 \to M$ in a contact manifold (M, ξ) is called *transverse* if $\gamma'(t)$ is transverse to $\xi_{\gamma(t)}$ for all $t \in S^1$. It can be shown that any curve can be made transverse by a C^0 small isotopy. Notice a transverse curve can be *positive* or *negative* according as $\gamma'(t)$ agrees with the co-orientation of ξ or not. We will restrict our attention to positive transverse knots (thus in this paper "transverse" means "positive transverse"). Given a transverse knot γ in (M, ξ) that bounds a surface Σ we define the *self linking number*, $l(\gamma)$, of γ as follows: take a non-vanishing vector field v in $\xi|_\gamma$ that extends to a non-vanishing vector field in $\xi|_\Sigma$ and let γ' be γ slightly pushed along v. Define

$$l(\gamma, \Sigma) = I(\gamma', \Sigma),$$

where $I(\cdot, \cdot)$ is the oriented intersection number. There is a nice relationship between $l(\gamma, \Sigma)$ and the singularities of the characteristic foliation of Σ. Let $d_\pm = e_\pm - h_\pm$ where $e_\pm$ and $h_\pm$ are the number of $\pm$ elliptic and hyperbolic points in the characteristic foliation Σ_ξ of Σ, respectively. In [Be] it was shown that

$$l = d_- - d_+.$$

When ξ is a *tight* contact structure and Σ is a *disk*, Eliashberg [E4] has shown, using the elimination lemma, how to eliminate all the positive hyperbolic and negative elliptic points from Σ_ξ (cf. [Gi1]). Thus in a tight contact structure $l(\gamma, \Sigma)$ is always negative.

3. Singular Foliations on the 2-Skeleton

The lens space $L(p,q)$ is, by definition, the union of two solid tori V_0 and V_1 glued together by a map $\phi : \partial V_1 \to \partial V_0$ which in standard coordinates on the torus is given by the matrix

$$\begin{pmatrix} -q & p' \\ p & r \end{pmatrix},$$

where p' and r satisfy $-rq - pp' = -1$. (A standard basis is given by μ, the boundary of a meridianal disk, and λ, a longitude for T^2 given by the product structure on V_i and oriented so that $\mu \cap \lambda = 1$.) We will also use a CW decomposition of $L(p,q)$ obtained from V_0 and V_1 as follows: let C be the core curve in V_0, x a point on C, D a disk in $L(p,q)$ that intersects V_1 in a meridianal disk and whose boundary wraps p times around C, and $B = L(p,q) \setminus D$ a 3-ball. Now $L(p,q)$ can be written as

$$\{x\} \cup C \cup D \cup B.$$

We call D the *generalized projective plane* in $L(p,q)$. Note that if $p = 2$ so that $L(p,q)$ is $\mathbb{R}P^3$ then D is a copy of $\mathbb{R}P^2 \subset L(p,q)$. Given a contact structure ξ on $L(p,q)$ we would like to understand the characteristic foliation, D_ξ, on D. To this end we begin by isotoping C so that it is transverse to ξ. Throughout this paper we will take V_0 to be a small standard neighborhood of the transverse curve C. We now have the following proposition which says that ξ on all of $L(p,q)$ is determined by D_ξ.

Proposition 3.1. *Let L_0 and L_1 be two copies of $L(p,q)$. Let ξ_i be a tight oriented contact structure on L_i and D_i the generalized projective plane in L_i, $i = 0, 1$. Assume that the 1-skeleton C_i of D_i is transverse to ξ_i. If a diffeomorphism $f : L_0 \to L_1$ may be isotoped so that it takes $(D_0)_{\xi_0}$ to $(D_1)_{\xi_1}$ then it may be isotoped into a contactomorphism.*

Proof. A slight modification of the standard proof that the characteristic foliation on a surface determines the contact structure on a neighborhood of the surface can be used to isotope f into a contactomorphism in a neighborhood of D_0. We can then use Eliashberg's characterization of tight contact structures on the 3-ball, Theorem 2.2, to isotope f into a contactomorphism on all of L_0, since $L_0 \setminus D_0$ is a 3-ball and f is already a contactomorphism in a neighborhood of its boundary. ☐

In the remainder of this section we derive a standard form for the characteristic foliation on the two skeleton $D \subset L(p,q)$ depending only on the homotopy class of ξ.

3.1. **The Euler class and the Singular Foliation.** We have already arranged that C is transverse to ξ; thus, if V_0 is a sufficiently small tubular neighborhood of C, the curve $\gamma = \partial V_0 \cap D$ will also be a transverse curve. It will be helpful to keep in mind that γ is homotopic to pC. We will interchangeably think of D as an embedded disk with boundary γ and the generalized projective plane in $L(p, q)$ (note this should cause no confusion since one uniquely determines the other).

As discussed in Section 2 the self-linking number $l = l(\gamma, D)$ of γ is related to the singularities of D_ξ. We can also relate $l = l(\gamma, D)$ to the homotopy class of ξ. To state this relation we must begin by recalling the definition of a $\mathbb{Z}_2$-refinement of the Euler class of ξ. The Euler class $e = e(\xi)$ is an even element of $H^2(L(p, q); \mathbb{Z}) = \mathbb{Z}_p$. If p is odd then then there is a unique element c in $H^2(L(p, q); \mathbb{Z})$ such that $2c = e$. However, if p is even then there are precisely two elements in $H^2(L(p, q); \mathbb{Z})$ that can be thought of as half of e. We cannot naturally associate one of these elements to ξ, however once a spin structure is fixed on $L(p, q)$ we can. Thus the $\mathbb{Z}_2$-refinement $\Gamma(\xi)$ of $e(\xi)$ is actually a map

$$\Gamma(\xi) : \mathrm{Spin}(L(p, q)) \to G,$$

where $\mathrm{Spin}(L(p, q))$ is the group of spin structures on $L(p, q)$ and $G = \{x \in H_1(L(p, q); \mathbb{Z}) : 2x = PD(e(\xi))\}$ (PD means Poincaré dual). For a general discussion of this invariant see [G] and for an exposition of spin structures see [GS].

Recall, if p is odd then $L(p, q)$ has a unique spin structure and if p is even there are precisely two and they may be distinguished as follows: given a spin structure on $L(p, q)$ there is a naturally induced spin structure on $L(p, q) \times [0, 1]$ and one may ask if this spin structure extends over a 4-dimensional 2-handle added to the curve C in $L(p, q) \times \{1\}$. One of the spin structures on $L(p, q)$ will extend over a 2-handle added with framing 0 and one will not. Note when p is odd $\Gamma(\xi)$ is determined by the element in $H^2(L(p, q); \mathbb{Z})$ that is half of $e(\xi)$ thus we refer to $\Gamma(\xi)$ as the *half-Euler class of* ξ. In general, $\Gamma(\xi)$ clearly refines the Euler class since $2\Gamma(\xi)(s) = PD(e(\xi))$ for any $s \in \mathrm{Spin}(L(p, q))$.

The map $\Gamma(\xi)$ is one-to-one and for computational convenience we shall actually define it as a map from G to $\mathrm{Spin}(L(p, q))$. For an alternate definition and its relation to the one we give here, in particular its well-definedness, see [G, Et1]. To define $\Gamma(\xi)$ let v be a vector field in ξ with zero locus (counted with multiplicity) $2c$ where c is a smooth curve in $L(p, q)$. Note that the homology class of c is in G. The vector field v gives a trivialization of ξ on $L(p, q) \setminus c$ and hence a trivialization of $TL(p, q)$ on $L(pq) \setminus c$. This trivialization induces a spin structure on $L(p, q) \setminus c$, and finally, since v vanishes to order two on c we can extend

this spin structure over c to obtain a spin structure s' on all of $L(p,q)$. The map $\Gamma(\xi)$ will associate c and s'. Specifically we define

$$\Gamma(\xi)(s') = [c].$$

We can now state the relation between $l(\gamma)$ and $\Gamma(\xi)$.

Theorem 3.2. *Let s' be any spin structure on $L(p,q)$. If $p > 0$ is even, let s be the spin structure on $L(p,q)$ that does not extend over a 2-handle attached to C with framing 0. Then*

$$\Gamma(\xi)(s') \cdot D \equiv \frac{1}{2}(-l(\gamma) + q + p\Delta(s,s')) \mod p, \tag{1}$$

where $\Delta(s,s') = 0$ if $s = s'$ and 1 otherwise. If p is odd this formula also holds if we always take $\Delta(s,s')$ to be 1 when q is even and 0 when q is odd.

Remark 3.3. When p is odd $L(p,q)$ has a unique spin structure which, when q is even, is not the structure s described in the theorem. This explains then strange definition of $\Delta(s,s')$ for odd p.

Remark 3.4. Note that in terms of the Euler class we get

$$e(\xi)(D) \equiv (-l + q) \mod p. \tag{2}$$

This formula is easier to prove and would suffice for p odd.

Proof. We begin by constructing a vector field w in $\xi|_D$ and use w to compute the Euler class of ξ. Then we modify w to a vector field whose zeros all have multiplicity two, thus allowing us to compute Γ. On $D \cap V_1$ we let w be a vector field directing the characteristic foliation D_ξ. We cannot use the characteristic foliation to define w on $D \cap V_0$ since it will not be well defined along C. But it is not hard to find a vector field in $\xi|_{D\cap V_0}$ that agrees with w on $\xi|_{\partial(D\cap V_0)}$ and has exactly q zeros in V_0. (If p is even this vector field will define a spin structure in a neighborhood of C that will not extend over a 2-handle attached to C with framing 0.) When one uses w to compute the Euler class of ξ one gets $-l + q$.

We now must coalesce the zeros of w into zeros with multiplicity two. For simplicity assume that p is even since otherwise Γ is determined by e and we are already done. We can also assume that we have canceled all the negative elliptic points and positive hyperbolic points in D_ξ, thus there will be $n+1$ elliptic points and n hyperbolic points (see the remark following this proof). Writing down a model for $D \cap V_0$ one can explicitly write a vector field with $\frac{q-1}{2}$ zeros of multiplicity 2 and one zero of multiplicity 1. Moreover this vector field will agree

with w on C and $\partial(D \cap V_0)$. Let v be this vector field on V_0. The zero of multiplicity 1 in V_0 will be connected by a leaf to an elliptic point on $D \cap V_1$ and the other elliptic and hyperbolic points will all pair up along stable separatrices of the hyperbolic points. It is now not hard to explicitly write down a vector field in a neighborhood of the connecting leaves of these pairs that agree with w outside the neighborhood and has precisely one zero of multiplicity 2 on each seperatrix. This will allow us to define v on all of D. One can now extend v to all of $L(p, q)$ and as mentioned above the spin structure it induces is s. Thus we have $\Gamma(\xi)(s) \cdot D \equiv \frac{1}{2}(-l + q) \mod p$. The formula for the other spin structure follows from general properties of Γ (see [G]). □

Remark 3.5. Theorem 3.2 tells us that

$$l(\gamma) = q - 2(\Gamma(\xi)(s) \cdot D) + 2np. \tag{3}$$

Recall, when ξ is tight l must be negative, limiting the possibilities for n. Moreover we can cancel all the negative elliptic and positive hyperbolic singularities from the characteristic foliation D_ξ. Thus we have

$$e_+ + h_- = -q + 2(\Gamma(\xi)(s) \cdot D) + 2np. \tag{4}$$

We also know that $e_+ - h_- = 1$ since we can use the characteristic foliation D_ξ to compute the Euler characteristic of the disk D. In the remainder of this section we will simplify the characteristic foliation more, eventually showing $0 < e_+ + h_- < 2p$. Notice that this will uniquely determine e_+ and h_-. (Again, when p is odd and q is even one should use Equation (2) as discussed in Remark 3.4.)

3.2. **Making Stars.** The *graph of singularities* of D_ξ will be defined to be the union of all singular points and stable separatrices (i.e. points that limit to a hyperbolic singularity in forward time) in D_ξ. When talking of the graph of singularities we always assume that $e_- = h_+ = 0$. It is easy to see that the graph of singularities must be a tree, but one can say much more.

Lemma 3.6. *We can choose D so that the graph of singularities of D_ξ forms a star, i.e. there is one $(e_+ - 1)$-valent elliptic vertex, $(e_+ - 1)$ univalent elliptic vertices and exactly one hyperbolic singularity in the interior of each edge (see Figure 1).*

Proof. This is a special case of a lemma in [ML], though the proof there seems to be incomplete. Some of the ideas below are also reminiscent of ones appearing in Fraser's thesis [F], though in a different setting.

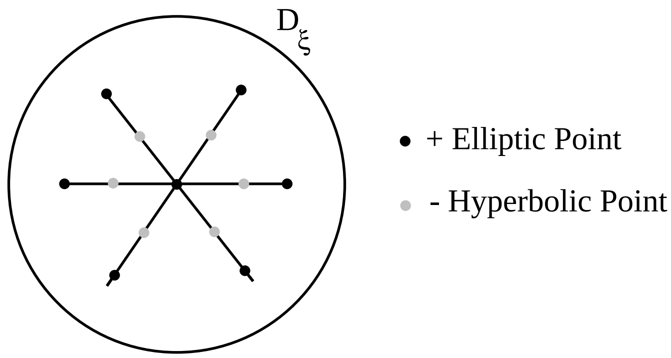

FIGURE 1. Singularities on D.

We will show how to isotope D to a disk D' with transverse boundary in ∂V_1, whose graph of singularities relates to D's as shown in Figure 2. Since the graph of singularities in D_ξ must be a tree, a sequence of such moves will clearly yield the conclusion of the lemma. Assume that part

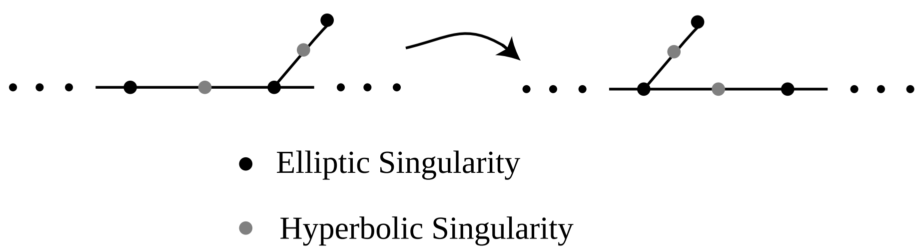

FIGURE 2. Change in Graph of Singularities.

of the graph of singularities in D_ξ is as shown on the left hand side of Figure 2. Let D_i be a subdisk of D with transverse boundary containing the graph of singularities in D_ξ. Let U be a neighborhood, diffeomorphic to an open ball, of D_i in V_1 and set $D_a = D \cap U$. Now $(U, \xi|_U)$ is a tight contact structure on $\mathbb{R}^3$, so the classification of contact structures on $\mathbb{R}^3$ [E5] implies there is a contactomorphism

$$f : (U, \xi|_U) \to (\mathbb{R}^3, \xi' = \{dz + x dy = 0\}).$$

In the following paragraph we show that there is a compactly supported isotopy of $f(D_a)$ to D'_a so that $(D'_a)_{\xi'}$ is related to D_ξ as shown in Figure 2. Then our desired disk is $D' = (D \setminus D_a) \cup f^{-1}(D'_a)$ (note the two pieces fit together since the above isotopy was compactly supported and the open disk $f(D_a)$ is properly embedded in $\mathbb{R}^3$).

We now need to prove our claim concerning the compactly supported isotopy of $f(D_a)$. To this end let Δ be a disk in the standard contact

structure on $\mathbb{R}^3$ whose characteristic foliation is related to D_ξ as indicated in Figure 3 (creating such a disk is an easy exercise). Note

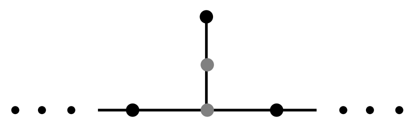

FIGURE 3. Characteristic Foliation on Δ.

that the characteristic foliation on Δ is unstable. Specifically, if we take a point p on the seperatrix connecting the two hyperbolic points and push it up, respectively down, Δ_ξ will look like the foliation indicated on the right side, respectively left side, of Figure 2. For later use we describe this isotopy in a standard model: let p be a point on the seperatrix, α a segment of the seperatrix containing p, and in $(\mathbb{R}^3, \xi')$ let p' be the point $(1,0,0)$ and α' the arc $\{(t,0,0) : \frac{1}{4} \leq t \leq 2\}$. Now there is a neighborhood O of α diffeomorphic to an open three ball and contactomorphic to a neighborhood O' of α' that does not intersect the yz-plane in $\mathbb{R}^3$. Denote this contactomorphism by g. Moreover, if we take O and O' sufficiently small we can assume that g takes $O \cap \Delta$ to the $O' \cap$ (xy-plane), α to α' and p to p'. One may now explicitly see that slightly pushing p' up or down (in the z-direction) will change the foliation as claimed. If O' is an ϵ neighborhood of α' (which we may assume by shrinking O and O') then choose a $\delta << \epsilon$ and push p' up slightly by an isotopy supported in the δ-ball around p'. Let Δ' be the image of Δ under the corresponding isotopy. Below we will find a compactly supported contactomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ taking $f(D_i)$ to Δ' and a neighborhood N of $f(D_i)$ to a neighborhood N' of Δ'. Assuming this for the moment we finish the proof of the lemma. By construction, $h(f(D_a))$ has the same characteristic foliation as D. To obtain the desired foliation we can isotope $h(f(D_a))$ in O to a disk we call D'_a so its characteristic foliation is as shown on the right of Figure 2 (this isotopy corresponds to pushing p' down a little in O').

The problem now is that D'_a may not be embedded if

$$K = h(f(D_a \setminus D_i)) \cap O \neq \emptyset.$$

Consider $K' = g(K)$ in our neighborhood O' of p' and by possibly shrinking N and N' we can assume that $g(N')$ is a δ' neighborhood of $g(\Delta' \cap O)$ where $\delta' << \delta$. Finally take m to be the maximum z coordinate in $g(\Delta' \cap O)$ (though not essential we can assume that $g(\Delta' \cap O)$ differs from the xy-plane by a small symmetric bump with a unique maximum m). Note that if $\delta' \geq m$ then D'_a will be embedded because

the isotopy from $h(f(D_a))$ to D'_a does not have to leave the neighborhood N'. So we assume that this is not the case. We can now choose a function $\psi : \mathbb{R} \to \mathbb{R}$ such that

1. $\psi(z) = z$ outside $[-\delta', m]$,
2. $\psi(z) < 0$ for $z < m - \delta'$,
3. $\frac{1}{2} \leq \psi'(z) \leq 1$ on $[-\delta', m - \delta']$ and
4. ψ is strictly increasing.

With ψ in hand we define a contactomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(x, y, z) = (f'(z)x, yf(z))$. So $\phi(K')$ clearly has the same characteristic foliation as K' (in particular it is non singular) and agrees with K' near the boundary of O'. Using properties 2. and 3. of ψ we see that $\phi(K')$ lies in O' and below the xy-plane. Thus if we replace K' by $\phi(K')$ (and of course make the corresponding alteration to D'_a) then we have eliminated all the possible self intersections of D'_a.

To finish the proof we now need to show that there is a compactly supported contactomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ taking a neighborhood N of $f(D_i)$ to a neighborhood N' of Δ'. Recall that as contact manifolds $\mathbb{R}^3 = S^3 \setminus \{q\}$, where S^3 has its standard tight contact structure and q is any point in S^3. So we actually construct a contactomorphism of S^3 that fixes a neighborhood of q. Since $f(D_i)$ and Δ' have identical characteristic foliations we can find a diffeomorphism h of S^3 fixing a neighborhood of q, taking $f(D_i)$ to Δ' and restricting to a contactomorphism on the neighborhoods N and U_0 of $f(D_i)$ and q, respectively. We may now isotop g into a contactomorphism on all of S^3 (this easily follows from the proof of Eliashberg's classification of tight contact structures on B^3 [E4] since g gives a map from the 3-ball $S^3 \setminus N$ to the 3-ball $S^3 \setminus g(N)$ preserving the characteristic foliation on their boundaries and restricting to a contactomorphism on the Darboux ball U_0). □

Remark 3.7. We now set up some notation that will be used through the rest of the paper. Assume that we have already arranged that the graph of singularities of D_ξ is a star. If $n + 1$ is the number of elliptic points in D_ξ then there are n hyperbolic points which we denote $h_1, \ldots, h_n$ numbered anti-clockwise. We also label the n-valent elliptic point e_0 and the other elliptic points e_i according to the hyperbolic point with which they share a flow line. A point x on the 1-skeleton C will break the boundary of D into p arcs, $B_1, B_2, \ldots, B_p$ (also numbered anti-clockwise). Notice that as we traverse the boundary of D anti-clockwise we will encounter both the end points of the unstable separatrices (the ones *not* shown in Figure 1) leaving h_1 then both leaving h_2 continuing in this fashion until we reach the end points coming

from h_n. We label the end points of the unstable separatrices leaving h_i as h_i^a and h_i^c so that h_i^a is anti-clockwise of h_i^c. Thus around the boundary of D we see the points $h_1^c, h_1^a, h_2^c, \ldots, h_n^c, h_n^a$ broken into p sets by the intervals B_i.

3.3. **Simplifying Stars.** So far we have arranged that the generalized projective plane D has the following properties:

- the one skeleton of D is transverse to ξ,
- there are no negative elliptic and no positive hyperbolic singularities in D_ξ, and
- the graph of singularities in D_ξ forms a star.

We can now get control of the number of branches in the graph of singularities.

Lemma 3.8. *If $e_+ > p$ then we can isotope D to D' so that D' is a generalized projective plane in $L(p, q)$ enjoying the above listed properties for D and $e_+(D') = e_+(D) - p$.*

Proof. We know the singularities of D_ξ form a star. Since we are assuming that $e_+ > p$ we know there are at least p edges in the star, hence there are $n \geq p$ hyperbolic points (one for each edge). Now using the notation of Remark 3.7 it is clear that if $n > p$ then at least one of the hyperbolic points, h_i say, has both its unstable separatrices exiting D through, say, B_j. This is also true when $n = p$. To see this choose the point x on C so that h_1^c is the closest point (in the anti-clockwise direction) to x and lying in B_1. Now if h_1 is not the point we are looking for then h_1^a must be leaving D through B_2 (or an interval further anti-clockwise). If we continue in this fashion and none of the points h_i for $i < p$ have their unstable separatrices leaving on the same B_j then h_p^a and h_p^c must both lie in B_p, thus proving our claim.

Having found this hyperbolic point h_i with both unstable separatrices leaving D through B_j we now describe an isotopy of D which will decrease e_+ by p. Note that the unstable separatrices of h_i separate a disk Δ from D which contains exactly one elliptic point e_i and has part of its boundary on B_j and the other part is made from the unstable separatrices of h_i. We use this disk Δ to guide our isotopy. Essentially we push (in an arbitrarily small neighborhood of Δ) the part of C that intersects Δ across the unstable separatrices of h_i. More precisely we can write down an exact model of our situation in V_0 and then isotope the interval $I = C \cap \Delta$ to the boundary of V_0 along Δ (see Figure 4). Next we isotope I across the unstable separatrices of h. One can write down a precise local model of Δ in which to do the isotopy. Note this is not an isotopy through transverse knots, as the one above, but at

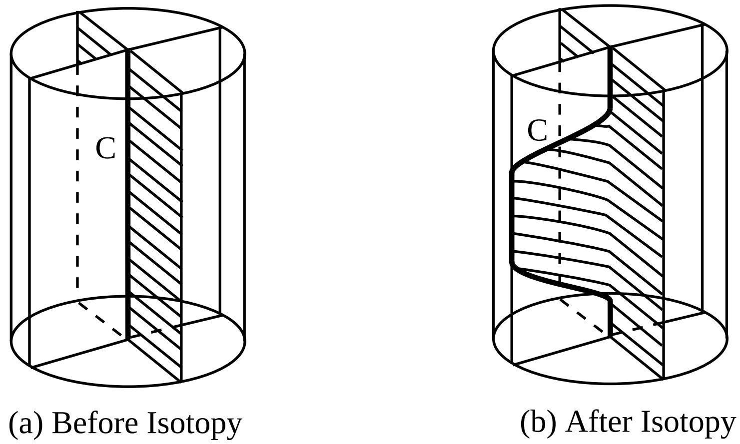

FIGURE 4. Model for V_0.

the end of the isotopy C will again be a transverse knot. To see what happens to the rest of D we consider the case when $p = 3$, the case for larger p being analogous. Near I we have Δ on one side of I and the other two branches of $D \cap V_0$, which we label A_1 and A_2, fanning out behind C (see Figure 5 (a)). We can assume that A_1 meets I so that

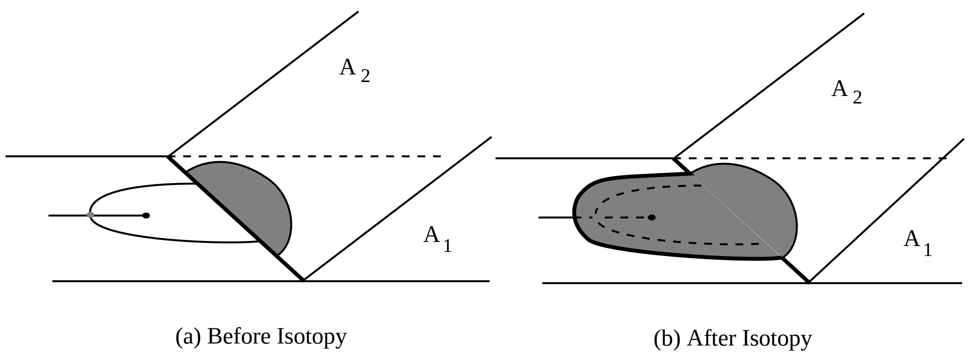

FIGURE 5. A_1 and A_2 near $C \cap V_0$.

it and Δ form a smooth surface. When we push I across the unstable separatrices of h_i we will transfer h_i and e_i from Δ to A_1. But the orientation Δ inherits after the isotopy, i.e. as a subset of A_1, is opposite the orientation it originally inherited from D (to see this consider the situation when $p = 2$ and we are dealing with a projective plane). We will also have to drag part of A_2 along with us through the isotopy (the gray part in Figure 5), but notice that we can drag it so that it is arbitrarily close to A_1 (see Figure 5 (b)). Thus the characteristic

foliation on the part of A_2 we dragged along is topologically equivalent to the foliation just transferred onto A_1, (since the foliation is structurally stable). In particular, this means that we have added an elliptic and a hyperbolic singularity to A_2. After isotoping C we have a new transverse curve C' and a new generalized projective plane D'. Using C' we also get a new Heegaard decomposition $L = V_0' \cup V_1'$ where V_0' is a small tubular neighborhood of C'.

We now claim that after canceling all the newly created negative elliptic and positive hyperbolic points on D' then the number of positive elliptic points, e_+', is $e_+ - p$. To see this note D' will essentially look like D with Δ removed in one place and $p-1$ copies of it glued on along subarcs of $B_2, \ldots, B_p$ (see Figure 6). The orientations on the copies of

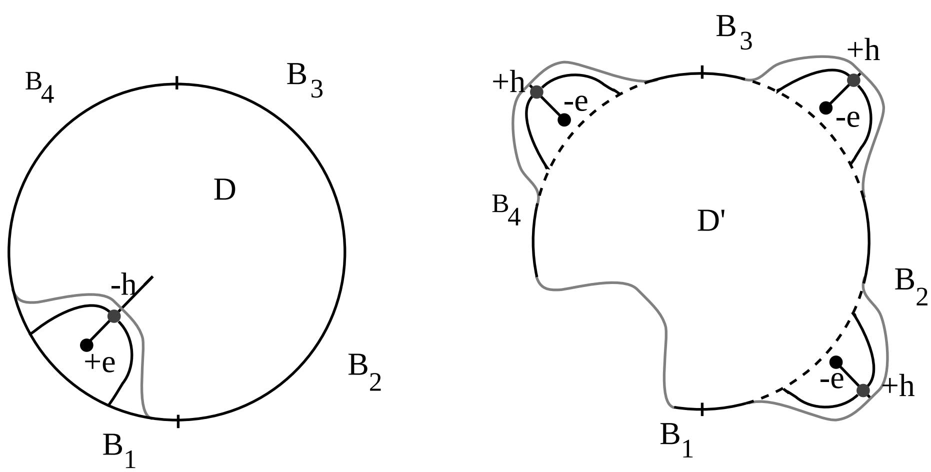

FIGURE 6. D and D' when $p = 4$.

Δ in D' will be opposite that of Δ in D. Thus D'_ξ has one less positive elliptic point and one less negative hyperbolic point than D_ξ but has $p-1$ more negative elliptic and positive hyperbolic points. Thus when we cancel the negative elliptic and positive hyperbolic points from D'_ξ we will have p fewer positive elliptic and negative hyperbolic point than D_ξ had. $\square$

Remark 3.9. Notice that the proof shows that, in a tight contact structure, if one is already in a minimal configuration, i.e. $e_+ \leq p$, then the unstable separatrices emanating from one hyperbolic point cannot both leave D along the same arc B_i no mater which point x is used to form the arcs B_i. This observation will be crucial in what follows.

4. Contact Structures on Lens Spaces

4.1. **The Existence of Tight Contact Structures.** The main technique for generating tight contact structures on 3-manifolds is to realize the 3-manifold at the "boundary" of a Stein manifold [E3, Gr] (we will think of our Stein manifolds as having boundaries), since the complex tangencies to the boundary induce a tight contact structure on the 3-manifold. For our purposes the exact definition of Stein manifold is unimportant (the curious reader is referred to [E2]) as the following theorem gives a useful characterization of these manifolds. But first we recall that a *Legendrian* knot in a contact 3-manifold (M, ξ) is a curve $\gamma : S^1 \to M$ with $\gamma'(t)$ in ξ_t for all $t \in S^1$. To a Legendrian knot γ bounding a surface Σ we can assign two invariants. The *Thurston-Bennequin invariant* of γ, $\mathrm{tb}(\gamma, \Sigma)$, is simply the integer given by the framing induced on γ by ξ (where Σ defines the zero framing). The *rotation number* of γ, $r(\gamma)$, is defined as follows: pick a trivialization of $\xi|_\Sigma$ and let T be a vector field tangent to γ. Define $r(\gamma, \Sigma)$ to be the degree of T with respect to this trivialization. The following theorem is implicit in Eliashberg's paper [E2]. For a complete discussion of this theorem see the paper [G] of Gompf.

Theorem 4.1. *An oriented 4-manifold X is a Stein manifold if and only if it has a handle decomposition with all handles of index less than or equal to 2 and each 2-handle is attached to a Legendrian circle γ with the framing on γ equal to $\mathrm{tb}(\gamma) - 1$. Moreover, the first Chern class $c_1(X)$ is represented by the cocycle*

$$c = \sum r(\gamma_i) f_{h_i},$$

where the sum is over the knots γ_i to which the 2-handles h_i are attached and f_{h_i} is the cochain that is 1 on core of h_i and 0 elsewhere.

One can use this to construct tight contact structures on every lens space $L(p, q)$ and compute their Euler class (since $e(\xi) = c_1(X)|_{\partial X}$). To do this assume p and q are both positive (this is no restriction) and let $r_0, r_1, \ldots, r_n$ be a continued fraction expansion of $-\frac{p}{q}$. The Kirby diagrams in Figure 7 both represent $L(p, q)$ (for more on Kirby diagrams see [GS]) and the diagram on the right can easily be made into an appropriate Legendrian link since all the surgery coefficients are integers less that 1. In general we can only use this construction to construct one tight contact structure on $L(p, q)$, but on $L(p, 1)$ we can do much better [E5]. If p is odd, then we can realize all non-zero elements of $H^2(L(p, 1); \mathbb{Z})$ as Euler classes of a tight contact structure. If p is even, then we can realize all non-zero elements of $H^2(L(p, 1); \mathbb{Z})$

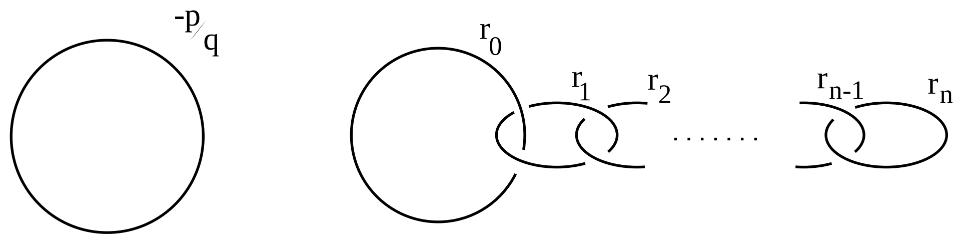

FIGURE 7. Two Kirby diagrams of $L(p,q)$.

as "$\Gamma(\xi)(s)$" (where s is as in Theorem 3.2) of a tight contact structure. We will see below that the "missing classes" above actually *cannot* be the Euler class (half Euler class) of a tight contact structure.

In [EG2] it was shown that one can do any Dehn surgery on unknots in S^3 (and some restricted surgeries on other knots) to obtain tight contact structures. Our main concern here is the existance of a tight contact structure realizing a particular half-Euler class.

Theorem 4.2 ([EG2])**.** *Let h be the 2-homology class determined by $h \cdot D = \frac{1}{2}(q+1) \mod p$. Any lens space $L(p,q)$ admits a tight contact structure with $e(\xi) = 2h$ if p is odd and $\Gamma(\xi)(s) = h$ if p is even.*

4.2. **Uniqueness and Non-Existence of Tight Contact Structures.** We begin by considering when the half-Euler class of a tight contact structure determines the structure.

Theorem 4.3. *Let $L(p,q), p > 0$, be a lens space and ξ_i, $i = 0, 1$, be two tight contact structures on $L(p,q)$. If*

$$\Gamma(\xi_i)(s') \cdot D = \pm\frac{1}{2}(q + 1 + p\Delta(s,s')) \mod p, \tag{5}$$

for $i = 0, 1$, where s, s' and Δ are as in Theorem 3.2, then ξ_0 and ξ_1 are contactomorphic.

Proof. When $\Gamma(\xi_i)(s') \cdot D = \frac{1}{2}(q+1+p\Delta(s,s')) \mod p$ we can use Formula 3 to see that $l = -1 + 2np$. Thus Lemma 3.8 allows us to arrange for D_{ξ_i} to have exactly one singular point which will have to be a positive elliptic point. We can actually think of these isotopies as ambient isotopies of $L(p,q)$. Thus we have isotoped the identity map to one which takes a generalized projective plane (in the domain lens space) with simple characteristic foliation to a generalized projective plane (in the range lens space) with simple characteristic foliation. We can further isotope our map so that it preserves the characteristic foliation on D. Thus Proposition 3.1 will produce the desired contactomorphism.

Every lens space $L(p,q)$ has an orientation preserving diffeomorphism that acts on $H^2(L(p,q))$ by multiplication by -1 [Bo]. This allows us to reduce the $\Gamma(\xi_i,s')\cdot D = -\frac{1}{2}(q+1+p\Delta(s,s')) \mod p$ case to the one above. □

Remark 4.4. Some lens spaces have other orientation preserving diffeomorphisms. Using these one might hope to find other homology classes supporting at most one tight contact structure; however, this does not seem to work since these diffeomorphisms permute the spin structures on $L(p,q)$. So the action of the diffeomorphism on cohomology coupled with the action on the spin structures conspire to prevent us from generalizing the above theorem.

This theorem simplifies when p is odd.

Corollary 4.5. *Let $L(p,q), p>0$, be a lens space and ξ_i, $i=0,1$, be two tight contact structures on $L(p,q)$. If p is odd and*

$$e(\xi_i)(D) = \pm(q+1) \mod p, \tag{6}$$

for $i=0,1$, then ξ_0 and ξ_1 are contactomorphic.

We can also prove that some cohomology classes cannot be realized by any tight contact structure.

Theorem 4.6. *Let ξ be a contact structure on $L(p,q), p>1$. If*

$$\Gamma(\xi)(s')\cdot D = \pm\frac{1}{2}(q-1+p\Delta(s,s')) \mod p, \tag{7}$$

where s, s' and Δ are as in Theorem 3.2, then ξ is overtwisted.

Proof. We begin by assuming that ξ is tight and proceed to find an overtwisted disk. In the $\Gamma(\xi,s')\cdot D = \frac{1}{2}(q-1+p\Delta(s,s')) \mod p$ case we have have p elliptic points according to Lemma 3.2 and Remark 3.5. Thus we can assume the graph of singularities of D_ξ is a star with $p-1$ branches. Using the notation of Remark 3.7 we can choose a point x on C so that h_1^c is closest (anti-clockwise) to x and lying in B_1. The unstable seperatrix containing h_1^c re-enters D after passing through C in $p-1$ arcs, which we denote $a_2,\dots,a_p$. We now claim that a_i ends at e_{i-1} for $i=2,\dots(p-1)$. To see this assume it is false and let i be the smallest index for which a_i does not end at e_{i-1}. If a_i ends anti-clockwise of e_{i-1} then both h_{i-1}^a and h_{i-1}^c must be on B_{i-1} contradicting Remark 3.9. See Figure 8. Now if a_i is clockwise of e_{i-1} then $h_{i-1}^a, h_{i-1}^c, \dots, h_p^a, h_p^c$ must lie in $B_i\cup\ldots\cup B_p\cup B_1$ with no two points coming from the same hyperbolic point lying in the same B_j. There are not, however, enough $B_j's$ for this; thus proving out claim.

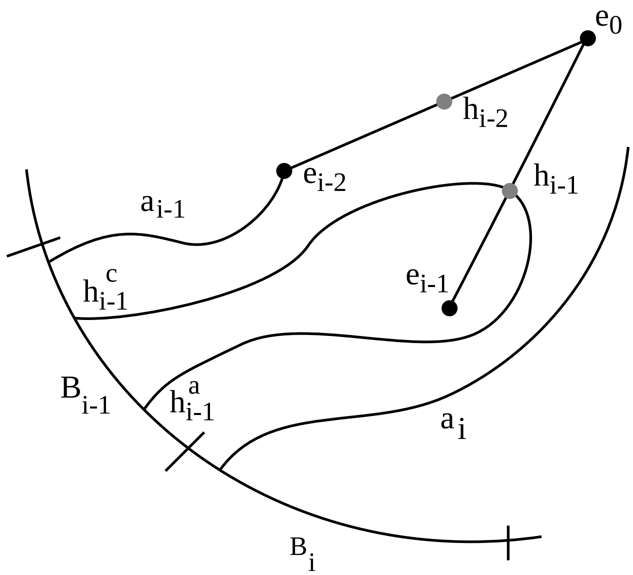

FIGURE 8. Impossible Configuration.

Arguing in a similar fashion with the rest of the separatrices we eventually see that the characteristic foliation must look like the one shown in Figure 9. In this picture we can explicitly find an overtwisted

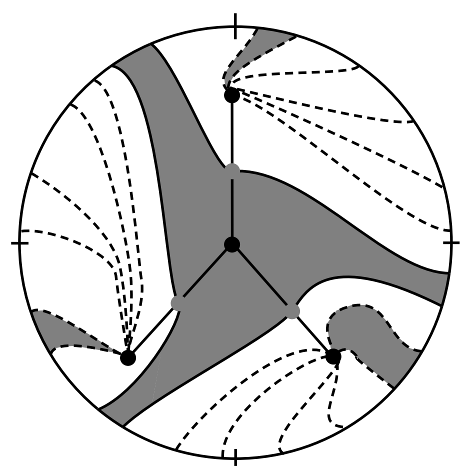

FIGURE 9. An Overtwisted Disk: dotted lines indicate arcs in the characteristic foliation corresponding to the continuation of unstable manifolds across C, e.g. the a_i's.

disk by canceling the singularities on the boundary of the shaded region in Figure 9.

Finally, if $\Gamma(\xi, s') \cdot D = -\frac{1}{2}(q - 1 + p\Delta(s, s')) \mod p$ then one uses the diffeomorphism discussed at the end of the proof of Theorem 4.3 to reduce to the above case. □

Again we have a simpler statement when p is odd.

Corollary 4.7. *Let ξ be a contact structures on $L(p,q), p > 1$. If p is odd and*

$$e(\xi)(D) = \pm(q-1) \mod p, \tag{8}$$

then ξ is overtwisted.

The above theorems do not provide a complete classification of contact structures on all lens spaces. The best general statement that can be made is given in the following theorem.

Theorem 4.8. *On any lens space $L(p,q)$ there is at least one class in $H^2(L(p,q))$ realized by a unique tight contact structure and at least one class that cannot be realized by a tight contact structure.*

Currently, contact structures are classified on $L(p,q)$ only when $p < 4$.

Theorem 4.9. *Classified up to isotopy*

1. *If $p = 0$ then $L(p,q) = S^1 \times S^2$ and there is a unique tight contact structure.*
2. *If $p = 1$ then $L(p,q) = S^3$ and there is a unique tight contact structure.*
3. *If $p = 2$ then $L(p,q) = \mathbb{R}P^3$ and there is a unique tight contact structure.*
4. *On $L(3,1)$ there are exactly two tight contact structures (one for each non zero element in $H^2(L(3,1);\mathbb{Z})$).*
5. *On $L(3,2)$ there is exactly one tight contact structure (realizing the zero class in $H^2(L(3,1))$).*

The first three statements were proved by Eliashberg [E4]. All but the first statement follow immediately from the theorems in this section. It is an interesting exercise to directly prove 4. by just considering D_ξ and not resorting to the diffeomorphism used in Theorem 4.3. Note that for all the examples mentioned in this theorem there is a unique contact structure up to contactomorphism but this is not always the case, as exemplified by $L(4,1)$ which has at least two tight contact structures up to contactomorphism.

4.3. **Finiteness Results.** Though work of Kronheimer and Mrowka [KM] indicates that tight contact structures exist in only finitely many homotopy classes of plain fields, it is not, in general, known if any given 3-manifold has a finite number of tight contact structures. There are examples of manifolds with infinitely many structures. For example Giroux [Gi2] and Kanda [K] have shown that T^3 has infinitely many tight contact structures. To show there were infinitely many structures on T^3, essential use was made of incompressible tori in T^3. One might hope that on atoroidal manifolds there are only finitely many tight contact structures. Thus lens spaces, being atoroidal, should have only a finite number of tight contact structures. This is indeed the case.

Theorem 4.10. *Any lens space admits only finitely many tight contact structures.*

Proof. On $L(p, q)$ there are between 1 and $p-1$ positive elliptic singularities. Once the number of positive elliptic singularities is determined the entire characteristic foliation D_ξ is determined by the cyclic ordering of the h_i^c's and h_i^a's along C and the grouping of these points in the B_i's. Since there are only a finite number of ways to order and group these points the proof is complete. □

Remark 4.11. Refining the analysis in this proof of the structure of the characteristic foliation on D one could derive a crude upper bound on the number of tight contact structures on a given lens spaces.

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